Scaling properties of granular materials

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Given an assembly of viscoelastic spheres with certain material properties, we raise the question how the macroscopic properties of the assembly will change if all lengths of the system, i.e. radii, container size etc., are scaled by a constant. The result leads to a method to scale down experiments to lab size.

Assume the dynamics of a certain granular system S is known. Will the dynamics change if we rescale all sizes by a constant factor α, i.e., $R_i' \equiv \alpha R_i$, but leaving the material properties unchanged [1]? If scaling affects the system properties, how do we have to modify the material properties to assure that the system S and the scaled system S' behave identically?

It is frequently desired to investigate large scale phenomena in granular systems experimentally, such as geophysical effects or industrial applications. To this end one has to rescale all lengths of the system to meet the restrictions of the laboratory size, i.e. big boulders in the original system become centimeter sized particles in the experiment. Of course, one wishes that the effects that occur in the large system occur equivalently in the scaled system too. With the assumption of viscoelastic particle deformation we will show that naive scaling will modify the properties of a granular system such that the original system and the scaled system might reveal quite different dynamic properties. To guarantee equivalent dynamical properties of the original and the scaled systems we have to modify the material properties in accordance with the scaling factor and we have to redefine the unit of time.

As another consequence of the scaling properties we claim that for numerical simulations of granular material it is not sufficient to provide relative data such as, e.g., to describe the container size in units of the particle diameter. We will show an example where the dynamics of a granular system changes significantly with system size, although all relative sizes are kept constant.

In a simple approximation, a granular system may be described as an assembly of spheres of radii R_i, $i = 1, \ldots, N$. If two particles of radii R_i and R_j at positions $\vec{r}_i$ and $\vec{r}_j$ touch, i.e., if $\xi_{ij} \equiv R_i + R_j - |\vec{r}_i - \vec{r}_j| > 0$, they feel an interaction force

$$\vec{F}_{ij} = F_{ij}^n \vec{n}_{ij} + F_{ij}^t \vec{t}_{ij}, \tag{1}$$

with the unit vector in normal direction $\vec{n}_{ij} \equiv (\vec{r}_j - \vec{r}_i)/|\vec{r}_j - \vec{r}_i|$ and the respective unit vector in tangential direction $\vec{t}_{ij}$. Eventually, external forces as, e.g., gravity, may also act on the particles.

The normal force F^n can be subdivided into elastic and dissipative parts $F^n = F_{el}^n + F_{dis}^n$ [2]. The elastic force for colliding spheres is given by Hertz's law [3]

$$F_{el}^n = \frac{2Y}{3(1-\nu^2)} \sqrt{R^{eff}} \xi^{3/2} \equiv \kappa \xi^{3/2}, \tag{2}$$

with $R^{eff} = R_i R_j / (R_i + R_j)$ and Y, ν being the Young modulus and the Poisson ratio. Equation (2) also defines the prefactor κ that we will need below.

The formulation of the dissipative part of the force F_{dis}^n depends on the mechanism of damping. Here we will focus on viscoelastic damping that is the most simple assumption for dissipatively colliding bodies [4]. It implies that the elastic part of the stress tensor is a linear function of the deformation tensor and the dissipative part of the stress tensor is a linear function of the deformation rate tensor. It is valid if the characteristic velocity (the impact rate g) is much smaller than the speed of sound c in the material and the viscous relaxation time τ_{vis} is much smaller than the duration of the collisions τ_c [5]. The range of the viscoelastic model is, hence, limited from both sides: the collisions should not be too fast to assure $g \ll c$, $\tau_{vis} \ll \tau_c$, and not too slow to avoid influences of surface effects as adhesion. For viscoelastically colliding spheres the dissipative part of the normal force reads [5–7]

$$F_{dis}^n = A \frac{d\xi}{dt} \frac{d}{d\xi} F_{el}^n = \frac{3}{2} A \kappa \sqrt{\xi} \frac{d\xi}{dt}. \tag{3}$$

The dissipative material constant A is a function of the viscous constants $\eta_{1/2}$, the Young modulus Y and the Poisson ratio ν (for details see [5]),

$$A = \frac{1}{3} \frac{(3\eta_2 - \eta_1)^2}{3\eta_2 + 2\eta_1} \frac{(1-\nu)(1-2\nu)}{Y\nu^2}. \tag{4}$$

Combining the forces (2) and (3) one obtains the equation of motion

$$\frac{d^2\xi}{dt^2} + \frac{\kappa}{m^{eff}} \left(\xi^{3/2} + \frac{3}{2} A \sqrt{\xi} \frac{d\xi}{dt} \right) = 0, \tag{5}$$

with $m^{eff} = m_i m_j / (m_i + m_j)$ and with the initial conditions $\xi|_{t=0} = 0$ and $d\xi/dt|_{t=0} = g$.

In dimensionless variables, $\hat{\xi} \equiv \xi/\xi_0$, $\tau \equiv t/\tau_0$, Eq. (5) reads [8]

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$$\frac{d^2\hat{\xi}}{d\tau^2}+\frac{5}{4}\hat{\xi}^{3/2}+\frac{3}{2}\left(\frac{5}{4}\right)^{3/5}A\left(\frac{\kappa}{m^{\mathrm{eff}}}\right)^{2/5}g^{1/5}\sqrt{\hat{\xi}}\frac{d\hat{\xi}}{d\tau}=0, \quad (6)$$

with $\hat{\xi}|_{\tau=0}=0$ and $d\hat{\xi}/d\tau|_{\tau=0}=1$. The characteristic length ξ_0 is the maximal compression for the equivalent undamped (elastic) problem that can be found by equating the kinetic energy of the impact $m^{\mathrm{eff}}g^2/2$ and the elastic energy at the instant of maximal compression $2\kappa\xi_0^{5/2}/5$. As characteristic time τ_0 we define the time in which the distance between the particles changes by the characteristic length just before the collision starts [9]

$$\xi_0\equiv\left(\frac{5}{4}\frac{m^{\mathrm{eff}}}{\kappa}\right)^{2/5}g^{4/5}, \quad \tau_0\equiv\xi_0/g. \quad (7)$$

The only term in Eq. (6) that depends explicitly on the system size and on material properties is the prefactor in front of the third term. If the scaling procedure affects the value of this term it will change the dynamics of the system. To ensure identical behavior of the scaled system, however, besides the identity of this prefactor, further requirements have to be met that will be discussed below.

Expanding our abbreviations we obtain

$$A\left(\frac{\kappa}{m^{\mathrm{eff}}}\right)^{2/5}g^{1/5}=(2\pi)^{-2/5}AF(R_i,R_j)g^{1/5}\left(\frac{Y}{\rho(1-\nu^2)}\right)^{2/5}, \quad (8)$$

with ρ being the material density. The function $F(R_i,R_j)$ collects all terms containing R_i and R_j:

$$F(R_i,R_j)\equiv\frac{[R_iR_j/(R_i+R_j)]^{1/5}}{[R_i^3R_j^3/(R_i^3+R_j^3)]^{2/5}} \quad (9)$$

Scaling the radii by α, the function F scales $F(R_i',R_j')=F(\alpha R_i,\alpha R_j)=\alpha^{-1}F(R_i,R_j)$. Obviously, simple scaling of the system in general affects the prefactor Eq. (8) already via the scaling properties of $F(R_i,R_j)$, hence, in general the original system and the system where the lengths have been scaled by a factor α differ in their dynamic properties. More explicitly, one can show that naively scaling the system by a factor $\alpha<1$ will lead to a comparatively more damped dynamics.

To provide equivalent dynamical properties of the systems, therefore, we have to modify the material properties in a way to assure that the equations of motion of both systems are equivalent, which in turn assures that the prefactors (8) of the original system and the scaled system are identical.

One of the few things that cannot be modified in experiments with reasonable effort is the constant of gravity G. That implies that going from S to S' not only G but all other accelerations must remain unaffected too,

$$\left(\frac{d^2x}{dt^2}\right)'=\frac{d^2(\alpha x)}{d(t')^2}=\frac{d^2x}{dt^2}, \quad (10)$$

yielding $t'=\sqrt{\alpha}t$. Hence, scaling all lengths $x'=\alpha x$ implies that times scale as $t'=\sqrt{\alpha}t$ if we require that the gravity constant stays unaffected. Thus, the clock in the scaled sys-

tem S' must run by a factor α faster (or slower) than the clock in the original system. In other words, if in the original system we observe a phenomenon at time $t=100\,\mathrm{s}$, we will find the same effect in the scaled system at time $t'=\sqrt{\alpha}100\,\mathrm{s}$. Scaling of time is a direct consequence of scaling the lengths if the constant of gravity has the same value in both systems.

In the scaled system the equation of motion of a particle contact reads

$$\frac{d^2\xi'}{dt'^2}+\frac{\kappa'}{(m^{\mathrm{eff}})'}(\xi')^{3/2}+\frac{3}{2}A'\frac{\kappa'}{(m^{\mathrm{eff}})'}\sqrt{\xi'}\frac{d\xi'}{dt'}=0. \quad (11)$$

If we apply our scaling relations that were introduced above, i.e.,

$$\xi'=\alpha\xi, \quad \frac{d\xi'}{dt'}=\sqrt{\alpha}\frac{d\xi}{dt}, \quad \frac{d^2\xi'}{dt'^2}=\frac{d^2\xi}{dt^2}, \quad (12)$$

we obtain

$$\frac{d^2\xi}{dt^2}+\alpha^{3/2}\frac{\kappa'}{(m^{\mathrm{eff}})'}\xi^{3/2}+\frac{3}{2}\alpha A'\frac{\kappa'}{(m^{\mathrm{eff}})'}\sqrt{\xi}\frac{d\xi}{dt}=0. \quad (13)$$

Comparing with Eq. (5) we find the conditions to assure identity of the equations of motion,

$$\frac{\kappa'}{(m^{\mathrm{eff}})'}=\alpha^{-3/2}\frac{\kappa}{m^{\mathrm{eff}}}, \quad A'=\sqrt{\alpha}A. \quad (14)$$

Using the definitions of κ [Eq. (2)] and m^{eff} yields finally

$$\left(\frac{Y}{\rho(1-\nu^2)}\right)'=\alpha\frac{Y}{\rho(1-\nu^2)}, \quad A'=\sqrt{\alpha}A. \quad (15)$$

If we choose material constants that obey Eqs. (15) we will obtain the original equation of motion after scaling the system back to its original size, i.e., both systems are equivalent.

When we incorporate the tangential force F^t into the analysis, of course, we have to require that this force scales in the same way as any other force, namely,

$$\frac{(F^t)'}{(m^{\mathrm{eff}})'}=\frac{F^t}{m^{\mathrm{eff}}}, \quad (16)$$

given that accelerations are invariant under scaling. This requirement has to be met by appropriately scaling the material constants, particularly the friction constant, resulting in an additional scaling equation. Its form depends on the underlying friction model, i.e., on the functional dependence of the tangential force on the geometry and the material properties as well as on the compression and the relative velocity. For instance, if we assume the most simple tangential force law

$$F^t=\mu F^n, \quad (17)$$

with μ being the friction coefficient we can conclude that the friction coefficient has to be invariant with respect to scaling

$$\mu'=\mu. \quad (18)$$

TABLE I. Material parameters used in the simulations.

Samples	Original	Scaled I	Scaled II
Plane length	10 m	62.5 cm	160 m
Particle diameter	10 cm	6.25 mm	160 cm
ρ	2 g/cm^3	2 g/cm^3	2 g/cm^3
$\frac{Y}{(1-\nu^2)}$	8 GPa	0.5 GPa	128 GPa
A	2×10^{-5} s	5×10^{-6} s	8×10^{-5} s
μ	0.1	0.1	0.1

FIG. 2. Density profile of the flow down an inclined plane. Thick solid line: original system together with scaled systems I and II ($\alpha_I = 1/16$, $\alpha_{II} = 16$, correct scaling, parameters see Table I). The curves collapse precisely to one single line. Dashed line: scaled system II, $\alpha = 1/16$ (incorrect scaling), dot-dashed line: scaled system, $\alpha = 16$ (incorrect scaling). Thin-solid line: system incorrectly scaled by $\alpha = 49$. For the scaled systems, the horizontal axis has been multiplied by 16 (I), 1/16 (II), and 1/49.

In literature, one can find a variety of different models that might be more realistic for the description of the dynamics of a granular material (for an overview see, e.g., [10]). Applying the same procedure for these laws will result in different scaling relations and, hence, in different conditions of the type Eq. (18). A thorough discussion of the scaling of more realistic tangential friction laws will be published elsewhere.

To demonstrate the derived scaling laws, we present results of a two-dimensional molecular dynamics simulations of stationary flow of 1000 particles down an inclined plane. In the ''original system'' the length of the plane is 10 m (periodic B.C.), its slope is 15°, and the particle diameters have been randomly chosen with an average of 10 cm. The material parameters are given in Table I. Particles are subjected to normal viscoelastic forces as well as tangential forces (of the simple type discussed above) that, although not very realistic, is enough for the purpose of showing the scaling properties. Figure 1 shows a snapshot of the flow.

The solid curve in Fig. 2 represents the density profile perpendicular to the inclined plane of the original system and of the correctly scaled systems I and II, taking into account the necessary modification of the material parameters according to Eqs. (15) and (18). The modified material parameters are summarized in Table I. It is not surprising that all three lines collapse perfectly, since in scaled variables the equations of motion of the original system and the correctly scaled system are identical. Therefore, particle trajectories and, hence, the density plot must be identical.

Figure 2 also shows two curves for the stationary density profile of the systems scaled by $\alpha = 1/16$ (system I) and $\alpha = 16$ (system II) where only the lengths have been scaled but erroneously the material parameters have been kept constant (incorrect scaling). These curves deviate from the profile of the original systems since the original and the scaled systems obey different equations of motion. As anticipated, the scal-
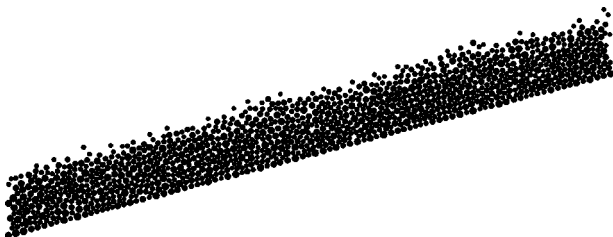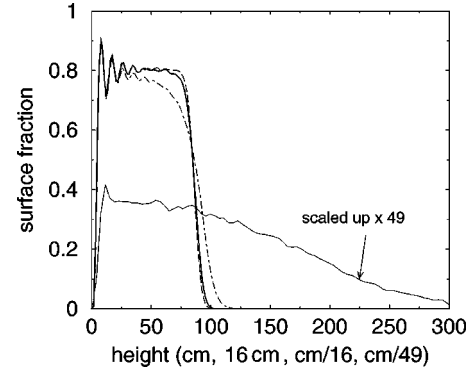
ing by $\alpha = 16$ only applied to lengths leads to a more elastic dynamics, i.e., comparatively more dilute flow (dot-dashed curve in Fig. 2). This effect is amplified when larger scaling factors are applied ($\alpha = 49$), where in twice the simulation time, no steady state is found while the density profile keeps relaxing towards more dilute states. The instantaneous density profile is shown as a thin-solid curve in Fig. 2, as an indication that incorrect scaling can ultimately lead to completely deviated results.

The simulations show that the simple length scaling of a granular system by a constant factor α changes the dynamical properties significantly, if the material parameters are kept constant. In order to obtain identical results for a scaled system one has also to modify the material constants and redefine the unit of time. These necessary scaling relations are summarized in Table II.

The knowledge about these scaling relations offers the possibility to scale down real world systems, e.g., geophysical or industrial granular systems, to sizes where laboratory experiments can be performed. If one scales down such a granular system one has to replace the original material by a material that meets the scaling requirements discussed in the text.

We want to give an example: Assume in the original system one deals with steel spheres ($Y = 20.6\times10^{10}$ Nm^{-2},

TABLE II. The necessary scaling relations when transiting from a granular system (S) to a scaled system (S') having identical dynamic properties.

	Original system	Scaled system
All lengths	x	αx
Time	t	$\sqrt{\alpha} t$
Elastic constant	$\dfrac{Y}{\rho(1-\nu^2)}$	$\alpha\dfrac{Y}{\rho(1-\nu^2)}$
Dissipative constant	A	$\sqrt{\alpha} A$

FIG. 1. Snapshot of the granular flow down an inclined plane.

$\nu = 0.29$ and $\rho = 7700$ kg m^{-3}) of average radius $\bar{R} = 10$ cm and system size of $L = 10$ m. The property whose scaling behavior is known is $Y/(\rho(1 - \nu^2)) = 2.92 \times 10^7$ m^2 s^{-2}. One wishes to know (to measure) a certain value at time $t = 100$ s. In the lab we perform the experiment with an equivalent system of size $L' = 1$ m, i.e., we scale the system by the factor $\alpha = 0.1$, including all radii. From the scaling relations we see that we have to find a material whose scaled property is $Y'/(\rho'(1 - \nu'^2)) \approx 0.3 \times 10^7$ m^2 s^{-2}. From tables [11] we find that we can use plexiglass ($Y = 0.32 \times 10^{10}$ Nm^{-2}, $\nu = 0.35$ and $\rho = 1200$ kg m^{-3}) in order to obtain this value. Therefore, we have to perform the experiment with plexiglass spheres and have to measure the value of interest at time $t' = 31.6$ s.

One can imagine that not for all scaling factors α one will find a proper material, however, nowadays it is possible to manufacture materials that can meet demanding requirements, such as high softness along with a custom-designed damping constant.

The scaling properties have also consequences for molecular dynamics simulations of granular systems: namely, it is not sufficient in simulations to provide *relative* parameters such as the container size in units of the particle radius. As demonstrated, the result of a simulation (and, of course, also of a real world experiment) depends on *absolute* values.

[1] Here and in the following we mark all variables that describe the scaled system S' with a prime.

[2] For simplicity of notation we drop the indices ij of the variables that refer to pairs of particles.

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[4] More complicated mechanisms are plastic deformation or brittle deformation that are problematic for a theoretical investigation since the shape of the particles changes due to collisions. Therefore, the simultaneous assumption of plastic deformation and spherical shape of the particles is inconsistent. It is not even clear that the spherical shape of particles that undergo plastic deformation is conserved on average. See W. F. Busse and F. C. Starr, Am. J. Phys. **28**, 19 (1960).

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[9] Note that up to a numerical prefactor the time scale τ_0 is equal to the duration of the undamped collision [3], which would be an alternative (and equivalent) choice of the time scale.

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